Math 211- Autumn 2023 – Sample Homework Solutions

1. Define $f(x) = x^4 + x^3 - 5x - 3$. $f(0) = -3 < 0$, $f(3) = 90 > 0$. There exists at least one zero in the interval $[0, 3]$ by IVT.

2. (a) By MVT, there exists points in $[0, 2]$ satisfying:

$$f(2) = f(0) + 2f'(\xi) \tag{1}$$

or:

$$3\xi^2 - 6 = -1$$

yielding $\xi = \sqrt{5/3}$.

(b) The slope is given by:

$$m = \frac{f(2) - f(0)}{2 - 0} = \frac{-3 - 1}{2} = -2 \neq f'(\xi) = -1$$

3. (a) The 3rd degree Taylor expansion around $x = 1$ is given by:

$$p_3(x) = f(1) + f'(1)(x-1) + \frac{1}{2}f''(1)(x-1)^2 + \frac{1}{6}f'''(1)(x-1)^3 \tag{2}$$

where:

$$\begin{aligned} f(x) &= \frac{x}{x+3} \Rightarrow f(1) = \frac{1}{4} \\ f'(x) &= \frac{3}{(x+3)^2} \Rightarrow f'(1) = \frac{3}{16} \\ f''(x) &= \frac{2x}{(x+3)^3} - \frac{2}{(x+3)^2} \Rightarrow f''(1) = -\frac{3}{32} \\ f'''(x) &= -\frac{6x}{(x+3)^4} + \frac{6}{(x+3)^3} \Rightarrow f'''(1) = \frac{9}{128} \end{aligned} \tag{3}$$

thus the polynomial reads:

$$p_3(x) = \frac{1}{4} + \frac{3}{16}(x-1) - \frac{3}{64}(x-1)^2 + \frac{3}{256}(x-1)^3$$

(c)(d) $p_3(3/2) \approx 0.33349$ and $f(3/2) \approx 0.33333$, with absolute error being $e = |p_3(3/2) - f(3/2)| = -1.5667 \times 10^{-4}$.

4(a)(b) Let step size be $\Delta x = \frac{1}{2}$, we have ($N = 4$):

$$U_f(P) = \sum_{i=0}^{N-1} f(x_i)\Delta x = \frac{1}{2}(f(0) + f(1/2) + f(1) + f(3/2)) = \frac{1}{2} \cdot \frac{9}{2} = \frac{9}{4}$$

$$L_f(P) = \sum_{i=1}^{N} f(x_i)\Delta x = \frac{1}{2}(f(1/2) + f(1) + f(3/2) + f(2)) = \frac{1}{2} \cdot \frac{5}{2} = \frac{5}{4}$$

(a) The exact integral of f is:

$$\int_0^2 f(x)dx = 2 \in [5/4, 9/4] \tag{4}$$

Rubrics

None for optional homework. Comments provided by discretion.

Math 211- Autumn 2023 – Homework 1 Solutions

1. We perform Taylor expansion with integral remainder around $x_0 = 2$ up to fourth degree. Note that a k-th order expansion requires the $(k+1)$-th derivative of f to be **continuous** (this is more stringent than the Lagrange version). We have:

$$p_4(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(4)}(x_0)}{4!}(x - x_0)^4 + R_4(x) \tag{5}$$

where:

$$R_n(x) = \frac{1}{n!}\int_{x_0}^{x} f^{(n+1)}(t)(x - t)^n dt \tag{6}$$

The first few derivatives of $f(x) = x \ln x$ are as follows:

1: $\ln x + 1$
2: $1/x$
3: $-1/x^2$
4: $2/x^3$
5: $-6/x^4$

Therefore our polynomial is:

$$p_4(x) = 2\ln 2 + (\ln 2 + 1)(x - 2) + \frac{1}{4}(x - 2)^2 - \frac{1}{24}(x - 2)^3 + \frac{1}{96}(x - 2)^4 \tag{7}$$

with:

$$p_4(3) = 3\ln 2 + \frac{39}{32}$$

On the interval $[2, 3]$, we have:

$$R_4(x) = -\frac{1}{4}\int_2^x \frac{1}{t^4}(x - t)^4 dt = -4x\ln x - 2 + (4\ln 2 - \frac{10}{3})x + 3x^2 - \frac{1}{2}x^3 + \frac{1}{24}x^4$$

$$0 \leq R_4(x) \leq 12(\ln 2 - \ln 3) + \frac{39}{8} \approx 9.419 \times 10^{-3}$$

whereas the absolute error is:

$$|f(3) - p_4(3)| \approx 2.355 \times 10^{-3}$$

2. Suppose without loss of generality that $x_1 < x_2$. For $f(x_1) < f(x_2)$, we have:

$$\begin{aligned} c_1 f(x_1) &< c_1 f(x_2) \\ c_2 f(x_1) &< c_2 f(x_2) \end{aligned} \tag{8}$$

Furthermore:

$$(c_1 + c_2)f(x_1) < c_1 f(x_1) + c_2 f(x_2) < (c_1 + c_2)f(x_2) \tag{9}$$

or:

$$f(x_1) < \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2} < f(x_2)$$

which implies by intermediate value theorem that there exists some $\xi \in [x_1, x_2]$ such that $f(\xi) = \frac{c_1 f(x_1) + c_2 f(x_2)}{c_1 + c_2}$. The case where $f(x_1) \geq f(x_2)$ is analogous.

3. (a) By intermediate value theorem, it suffices to show the function:

$$f(x) = -e^x + x^3 - 4$$

has opposite signs when evaluated at $x = 2$ and $x = 4$. We have $f(2) = 4 - e^2 < 0$ and $f(4) = 60 - e^4 > 0$. Therefore there must exist $c \in [2, 4]$ such that $f(c) = 0$.

(b) We follow the bisection algorithm in Chapter 3 of [KC] with starting interval $[2, 4]$. In 3 steps, we have the following results:

Iter.	c_n	$f(c_n)$	$[a_n, b_n]$	$\lvert c_n - c_{n-1}\rvert$	$\frac{\lvert c_n - c_{n-1}\rvert}{\lvert c_n\rvert}$
0	-	-	$[2, 4]$	-	-
1	3	2.91	$[2, 3]$	-	-
2	1.5	-5.11	$[1.5, 3]$	1.50	1.00
3	2.25	-2.10	$[2.25, 3]$	0.75	0.33

(c) The outputs in 3 steps are as follows:

Iter.	a_n	$f(a_n)$	b_n	$f(b_n)$	$\lvert a_n - a_{n-1}\rvert$	$\frac{\lvert a_n - a_{n-1}\rvert}{\lvert a_n\rvert}$
0	2.000	-3.389	4	5.402	-	-
1	2.770	1.303	2.000	-3.389	0.770	0.278
2	2.557	-0.178	2.770	1.303	0.213	0.0833
3	2.583	-0.00444	2.557	-0.178	0.026	0.0100

4. We provide an example where we show Newton's method diverges. Possible questions include the following:

- If a function has multiple roots, which one does Newton's method converge to? Is it sensitive to initial starting points?
- For what functions can Newton's method fail to converge?
- What happens if the root is the inflection point?

Consider locating the root with $x = x_0 \neq 0$ on an interval $[-c, c], c > 0$ for $f(x) = x^{1/3}$, whose second derivative at $x = 0$ is undefined. Newton's iteration gives us:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{1}{3}x_n^{-2/3}} = x_n - 3x_n = -2x_n \tag{10}$$

which implies that the error is doubled at each iteration.

Rubrics

General Rules: 0 point will be given for missing or illegible solutions. Partial credits will be given by discretion for partial solutions. Custom point adjustments may be assigned on an individual basis, with explanations. Absence of comments/blank comments accompanied by full credit can be interpreted as "great work". Selected common mistakes are numbered for each question, and directly referenced in the provided comments of graded assignments. Significantly deviant numerical results often refer to those differing from reference solution by at least 1 order of magnitude, or nonconvergent results (for root-finding problems).

1. (12 points)

(a) (6 points)

1. (-3 points) Taylor expansion of incorrect order
2. (-2 points) Incorrect terms in Taylor expansion
3. (-1 point) Incorrect final numerical result

(b) (6 points)

1. (-2 points) Misinterpreted definition of $R_4(x)$, which is a function of x. t was integrated out, and should not remain in the final expression.
2. (-1 point) Significantly deviant numerical result in absolute error, approximated solution. In particular, final result does not bound absolute error.

2. (8 points)

Points subtracted with custom comments.

3. (12 points)

(a)

Points subtracted with custom comments.

(b)

(-2 points) for significantly deviant results or intermediate steps.

(c)

(-2 points) for significantly deviant results or intermediate steps.

4. (8 points)

Points awarded/subtracted by discretion.

Math 211- Autumn 2023 – Homework 2 Solutions

1. Let $\{x_n\}$ denote the sequence of midpoints computed by the bisection method of Chapter 1 [KC], and $\lim_{n\to\infty} x_n = x^*$. Then the following error estimate holds:

$$|x_n - x^*| \le 2^{-(n+1)}(b_0 - a_0) \tag{11}$$

   Then it suffices to solve for n:

$$2^{-(n+1)}(b_0 - a_0) \le M \tag{12}$$

   yielding:

$$n \ge -\log_2\left(\frac{M}{b_0 - a_0}\right) - 1 \tag{13}$$

   Let $a_0 = 50, b_0 = 63, M = 10^{-3}$, we obtain:

$$n \ge 12.666 \tag{14}$$

   Therefore at least 13 iterations is required.

2. (a) Define $f(x) = x^3 - 4 - e^x$, we have $f'(x) = 3x^2 - e^x$, then Newton's iteration gives:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 4 - e^{x_n}}{3x_n^2 - e^{x_n}} \tag{15}$$

   The first few iterations along with their absolute errors are as follows:

$$\begin{aligned} x_0 &= 4.00000, e_0 = 0.38842 \\ x_1 &= 4.81869, e_1 = 0.43027 \\ x_2 &= 4.52476, e_2 = 0.13634 \\ x_3 &= 4.40690, e_3 = 0.018480 \\ x_4 &= 4.38882, e_4 = 3.96173 \times 10^{-4} \end{aligned} \tag{16}$$

   which converged within 10^{-3} in 4 iterations.

   (b) The derivative around a neighborhood of the local maximum will be very small, causing the update in Newton's method to overshoot and not converge to the desired root x^*, or the convergence rate is too slow. To mitigate this issue, one naive solution is to restrict the magnitude of update $\left|\frac{f(x_n)}{f'(x_n)}\right|$. If the function has more differentiability, Householder methods can be considered.

3. One of the methods to find a local maximum/minimum is to solve $f'(x) = 0$ using Newton's method, which enjoys second order convergence (if Newton's method does convergence). We have:

$$f(x) = x^2 \ln x, f'(x) = x + 2x\ln x, f''(x) = 2\ln x + 3 \tag{17}$$

   Suppose we initialize $x_0 = 1$. The first few iterations of Newton's method are as follows (rounded to 4 digits):

$$\begin{aligned} x_1 &= 0.6667, f'(x_1) = 0.12612 \\ x_2 &= 0.6091, f'(x_2) = 0.0051495 \\ x_3 &= 0.6065, f'(x_3) = -6.1318 \times 10^{-5} \\ x_4 &= 0.6065 \end{aligned} \tag{18}$$

4. See attached file `hw2p2_sol.ipynb`.

Rubrics

Common mistakes are documented below and directly referenced in Canvas comments. Custom point adjustments are given at discretion with comments. Absence of comments accompanied by full credit can be interpreted as "great work". If you submitted a Jupyter notebook, comments are left in the Canvas comment box.

Please group the PDF version of your Jupyter notebook and the write-up portion as one file for future homework assignments!

1. (10 points) One may use either relative error or absolute error as a metric. Full credits are given to either solutions.
   - (a) (-5) incorrect application of error upper bound (see Theorem 1 of Section 3.1 in [KC]), n should have a lower bound.
   - (b) (-2) Did not round n up to a whole number / fractional result.
   - (c) (-2) Correct application of theorem, but incorrect final result (differ by more than 1).
2. (12 points)
   (a) (6 points) Solutions to this part are evaluated heuristically. For Newton's method, one should observe doubling of digits of accuracy. Points are generally subtracted if one does not show all steps.
   (b) (6 points) Heuristic evaluation.
3. (10 points) 5 points for Numerical results; 5 points for reason of choosing algorithm. Points are assigned heuristically based on each individual solutions.
4. (8 points) 2 points each part (for a total of 4 parts). Points evaluated on individual basis, general mistakes include:
   - (-2) divergent numerical results
   - (-2) missing solutions
   - (-2) incorrect usage of `plt.plot()` (formatting is not considered)

Math 211- Autumn 2023 – Homework 3 Solutions

1. To establish convergence, we first show that f is a contractive map. For any $x \in \mathbb{R}$, we have:

$$|f'(x)| = \left|\frac{2x}{(1+x^2)^2}\right| \tag{19}$$

and we investigate the maximum value attainable of $|f'(x)|$. Setting $f''(x) = 0$ yields:

$$\frac{8x^2}{(1+x^2)^3} - \frac{2}{(1+x^2)^2} = 0 \tag{20}$$

from which we can compute:

$$6x^2 = 2 \Rightarrow x = \pm\frac{1}{\sqrt{3}} \tag{21}$$

To be precise, one can compute $f'''(x)$ and plug in these values to verify that one is a minimizer, and one is a maximizer of $f'(x)$ (this should also be apparent since $f'(x)$ is an odd function). In particular,

$$c = \max_{x\in\mathbb{R}} |f'(x)| = \left|\frac{2/\sqrt{3}}{(1+1/3)^2}\right| = \frac{2}{\sqrt{3}} \cdot \frac{9}{16} = \frac{3\sqrt{3}}{8} \approx 0.64952 < 1 \tag{22}$$

This implies $f(x)$ is a contraction (by MVT, one have $|f(x) - f(y)| \leq |f'(\xi)| \cdot |x-y| \leq \frac{1}{2}c \cdot |x-y|$ for some $\xi \in \mathbb{R}$). Since $\mathbb{R}$ is a complete metric space, contractive mapping theorem implies any starting point x_0 will yield a unique solution to $x^* = f(x^*)$. By explicit computations, that solution will be the solution of:

$$x = f(x) \Rightarrow x = \frac{1}{2} \cdot \frac{1}{1+x^2} \Rightarrow x^3 + x - \frac{1}{2} = 0 \tag{23}$$

One can solve the cubic equation by hand, or use a calculator to compute a numerical result. The final answer should be close to 0.424.

2. In all of the following cases, we consider the general form $x_{n+1} = F(x_n)$ and investigate the order of convergence as defined on Page 104 of [KC], Error Analysis. Let the fixed point $s = 6^{1/15}$, we check if successive orders of derivatives of F are zeros at s.

   (a)

$$F(x) = x - \frac{x^{15}-6}{15x^{14}}, F'(x) = \frac{14}{15} - \frac{64}{15}x^{-15}, F''(x) = 64x^{-16} \tag{24}$$

   therefore $F'(s) = 0$, $F''(s) \approx 9.466$. Therefore the convergence is of second order, since $F'(x)$ vanishes at $x = s$.

   (b)

$$F(x) = \frac{70}{73}x + \frac{18}{73}x^{-14}, F'(x) = \frac{70}{73} - \frac{252}{73}x^{-15} \tag{25}$$

   such that $F'(s) \approx 0.38356 < 1$. Therefore the convergence is of first order.

   (c)

$$F(x) = 6^{1/14}x^{-1/14}, F'(x) = -\frac{6^{1/14}}{14}x^{-15/14} \tag{26}$$

   and $|F'(s)| \approx 0.071428 < 1$, convergence at first order.

   Comparing the three methods above the convergence speed would be ranked as the following:

$$(a) \geq (c) \geq (b) \tag{27}$$

3. (a) Following Horner's algorithm, we first arrange the coefficients of the original polynomial, from highest to lowest power:

$$\begin{bmatrix} 1 & 0 & -4 & 1 & -5 & 3 \end{bmatrix} \tag{28}$$

   Let $b_{n-1} = a_n = 1$, at every step, we compute:

$$b_{n-1} = a_n + x_0 \cdot b_n \tag{29}$$

   where x_0 is the input. Using this rule, we successively obtain:

$$\begin{aligned} b_4 &= 1 \\ b_3 &= 0 + 3 \cdot 1 = 3 \\ b_2 &= -4 + 3 \cdot 3 = 5 \\ b_1 &= 1 + 5 \cdot 3 = 16 \\ b_0 &= -5 + 16 \cdot 3 = 43 \\ b_{-1} &= 3 + 3 \cdot 43 = 132 = f(3) \end{aligned} \tag{30}$$

   (b) Example 5 of Page 114 in [KC] provides an illustrative example. We successively reduce the order of the polynomial by removing factors of $(x - x_0)$ and simultaneously evaluating the reduced polynomial at $x = x_0$, until n factors have been reduced, yielding the final coefficient c_n. A naive implementation of Horner's algorithm for finding coefficients of Taylor expansion requires $O(n+(n-1)+(n-2)+\cdots+1) \sim O(n^2)$ evaluations. The final result should yield coefficients (from highest to lowest order):

$$\begin{bmatrix} 1 & 15 & 86 & 235 & 298 & 132 \end{bmatrix} \tag{31}$$

(c) The derivative of f has the following coefficients (with powers of $(x-3)$),

$$\begin{bmatrix} 5 & 60 & 258 & 470 & 298 \end{bmatrix} \tag{32}$$

The first 5 steps of Newton's algorithm yields:

$$\begin{aligned} x_1 &= 2.55704, \epsilon_1 = |x_1 - x^*| = 0.47372 \\ x_2 &= 2.26759, \epsilon_2 = 0.18427 \\ x_3 &= 2.12309, \epsilon_3 = 0.039773 \\ x_4 &= 2.08567, \epsilon_4 = 0.0023565 \\ x_5 &= 2.08333, \epsilon_5 = 1.03851 \times 10^{-5} \end{aligned} \tag{33}$$

4. Solution attached in the next page.

Math 211-Homework3-PartII

October 23, 2023

1 Math 211 Autumn 2023

1.1 Homework 3 - PART II

Due: October 20, 2023

Submit in Canvas as a .ipynb file.

In this notebook, some parts of the code is given and you will be asked to complete given tasks. These tasks may be given in the code as comments (marked by # or in the form''comment''). Point values of each part is also specified in the corresponding cells.

[1]:
```
#first, packages are imported. matplotlib.pyplot is what we use to graph a␣
↪function
import numpy as np
import matplotlib.pyplot as plt
```

1.2 Newton's Method

1.2.1 Question 1 (2 points):

Let's first consider Newton's method. Complete missing parts in the code below.

[3]:
```
'''Missing parts in this code are given as comments.
Replace them with necessary info below so that when it runs
output should be iteration number, x1, and the absolute error in the estimate'''

def Newton(f,df,real_value,x0,epsilon,maxiter):
    for i in range(0,maxiter):
        x1=x0-f(x0)/df(x0)#complete code here (1 point)
        x0=x1
        abs_error=np.abs(x0-real_value)#complete code here (1 point)
        if abs_error<epsilon:
            break
        else:
            continue
    return (i,x1,abs_error)
```

[4]:
```
#Let's check our code with a well-known example.
def f(x):
    return x**2-4

def df(x):
    return 2*x


Newton(f,df,2,5,0.001,100)
```

[4]:
```
(3, 2.0000051812194735, 5.181219473460175e-06)
```

1.3 Fixed-Point Iteration

Rest of the notebook is about finding zeros of functions by using fixed point iteration method. This topic is throughly explained in section 2.2 of the 9th Edition of the Numerical Analysis book by Burden and Faires (published by Cengage in 2011).

[5]:
```
def FixedPoint(f,real_value,x0,epsilon,maxiter):
    for i in range(1,maxiter):
        x1=f(x0)
        x0=x1
        rel_error=np.abs(x1-real_value)/np.abs(real_value)
        if rel_error<epsilon:
            break
        else:
            continue
    return (i,x1,rel_error)
```

1.3.1 Question 2 (2 points):

Consider function $f(x) = x^2 - x - 5$. Write three functions that could be used to approximate the zero of f. At this point, we don't know if the functions would produce a sequence converging to the zero or not.

Write two functions that could be used in the fixed point algorithm, and converges to the zero of f. The first function is given for reference.

[7]:
```
real_value=2.79128784747792
```

[8]:
```
def f_1(x):
    return x-x**2+x+5
```

[17]:
```
def f_2(x):
    return 1+5/x#write second function (1 point)
```

[10]:
```
def f_3(x):
    return np.sqrt(x+5)#write third function (1 point)
```

1.3.2 Question 3 (2 points):

By using these functions, write codes below running FixedPoint function defined above. Use following values:

[11]:
```
x0=1
epsilon=1e-10 #equals 10**(-10)
max_iter=100
```

[18]:
```
#notice that this algorithm diverges
FixedPoint(f_1,real_value,x0,epsilon,max_iter)
```

```
---------------------------------------------------------------------------
OverflowError                             Traceback (most recent call last)
Input In [18], in <cell line: 2>()
      1 #notice that this algorithm diverges
----> 2 FixedPoint(f_1,real_value,x0,epsilon,max_iter)

Input In [5], in FixedPoint(f, real_value, x0, epsilon, maxiter)
      3 x1=f(x0)
      4 x0=x1
----> 5 rel_error=np.abs(x1-real_value)/np.abs(real_value)
      6 if rel_error<epsilon:
      7     break

OverflowError: int too large to convert to float
```

[19]:
```
FixedPoint(f_2,real_value,x0,epsilon,max_iter)
```

[19]: (53, 2.7912878476593006, 6.498100452469562e-11)

[14]:
```
#write code running FixedPoint algorithm above with f_3 and given values (1␣
↪point)
FixedPoint(f_3,real_value,x0,epsilon,max_iter)
```

[14]: (14, 2.7912878474101843, 2.4266789601080836e-11)

[1]:
```
#Which of these function iterations resulted in the faster convergence? Write␣
↪your answer below. (1 point)
```

The second function converged faster. In particular, let s denote the fixed point, $|f'(s)| \approx 0.179128$ for the second method, whereas $|f'(s)| \approx 0.64174$ for the first method. The first method is roughly 3.5825 times slower than the second, as expected.

1.4 Plot the sequence

1.4.1 Question 4 (2 points):

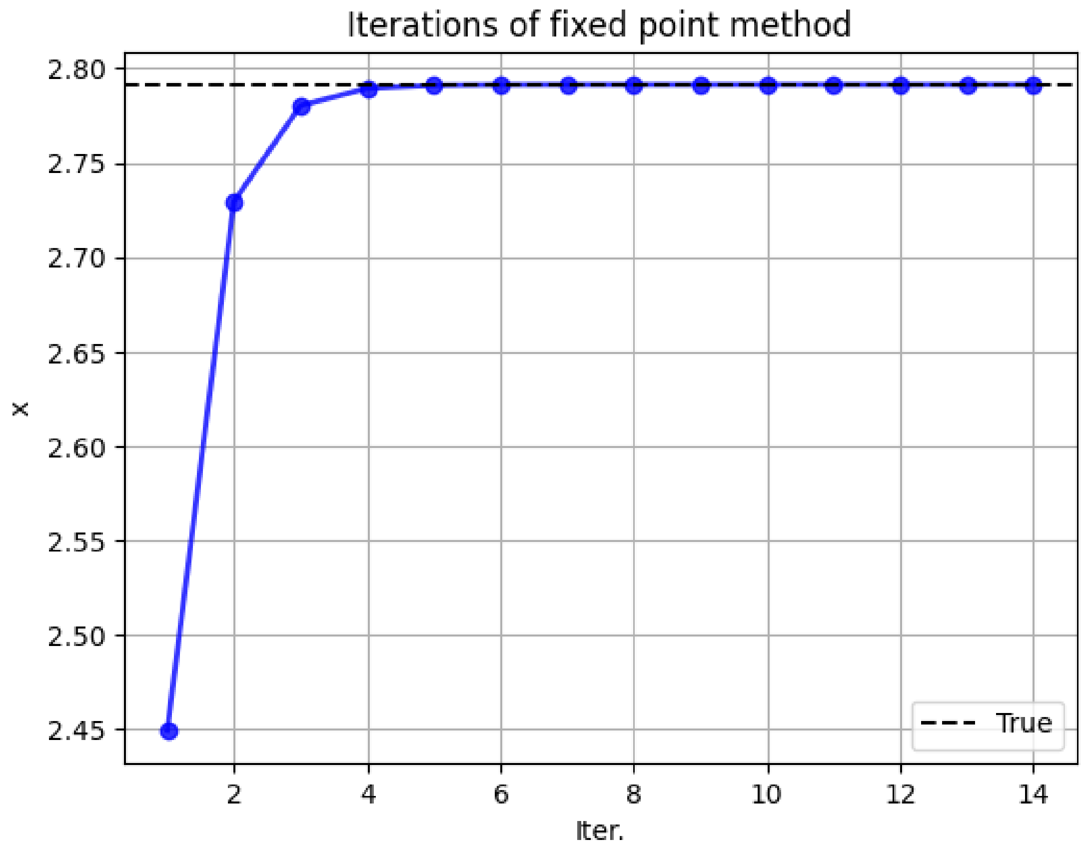
Now, modify the code for Fixed Point algorithm above so that code collects all iterated values in a list called sequence and gives the result as the last element in the output.

[20]:
```
def FixedPoint2(f,real_value,x0,epsilon,maxiter):
    sequence=[]
    for i in range(1,maxiter):
        x1=f(x0)
        x0=x1
        abs_error=np.abs(x1-real_value)
        #write a code that adds sequence elements to sequence (1 point)
        sequence.append(x0)
        if abs_error<epsilon:
            break
        else:
            continue
    return (i,x1,abs_error,sequence)
```

Now, let's plot the values that we have in the sequence produced for the fastest algorithm.

[32]:
```
'''write your code below, so that your plot produces the elements of the␣
 ↪sequence obtained by using the function that gives
the fastest convergence. (1 point)'''

#write your code here (1 point)
i, x_final, abs_error_final, seq =␣
 ↪FixedPoint2(f_3,real_value,x0,epsilon,max_iter)
plt.plot(np.arange(i)+1, seq, "-o", color="blue", alpha=0.8, lw=2.0);
plt.axhline(y=real_value, label="True", color="black", ls="--");
plt.title("Iterations of fixed point method");
plt.xlabel("Iter."); plt.ylabel("x");
plt.legend();
plt.grid(True);
plt.show()
```


Iterations of fixed point method
2.80
2.75
2.70
2.65
2.60
2.55
2.50
2.45
x
2
4
6
8
10
12
14
Iter.
True

```
[ ]:
```

Math 211- Autumn 2023 – Homework 4 Solutions

1. We have the system:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & 4 \\ 2 & 2 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} \tag{34}$$

a. We check that the leading principal minors are nonsingular:

$$\begin{aligned} &\det([1]) = 1, \qquad \det\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right) = -3 \\ &\det\left(\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}\right) = 2, \qquad \det\left(\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 2 & 3 \\ 1 & 1 & 1 & 4 \\ 2 & 2 & 4 & 5 \end{bmatrix}\right) = 4 \end{aligned} \tag{35}$$

b. The resulting matrix factorization should be:

$$A = \begin{bmatrix} 1 & & & \\ 2 & 1 & & \\ 1 & 1/3 & 1 & \\ 2 & 2/3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ & -3 & -4 & -5 \\ & & -2/3 & 5/3 \\ & & & 2 \end{bmatrix} \tag{36}$$

c. First invert L by forward substitution, then invert U by backward substitution. The final result should be:

$$x = \begin{bmatrix} -13/4 \\ -11/2 \\ 7/4 \\ 5/2 \end{bmatrix} \tag{37}$$

d. A does not admit a Cholesky decomposition because it is not symmetric (e.g. $a_{13} \neq a_{31}$).

2. To illustrate the differences, we compute the solution up to 3 decimals (with rounding). We have:

$$A = \begin{bmatrix} 0.003 & 5.000 \\ 1.000 & -2.000 \end{bmatrix}, b = \begin{bmatrix} 3.000 \\ 3.020 \end{bmatrix} \tag{38}$$

a. To carry out basic Gaussian elimination, we let R_1, R_2 denote row 1 and row 2 of A, and compute:

$$R_2 \leftarrow R_2 - R_1/0.003 \tag{39}$$

which yields the system:

$$\begin{bmatrix} 0.003 & 5.000 \\ 0 & -1668.667 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.000 \\ -996.980 \end{bmatrix} \tag{40}$$

Upon back-substitution, we obtain:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5.000 \\ 0.597 \end{bmatrix} \tag{41}$$

b. We swap R_1 and R_2, and perform Gaussian elimination again with 3-decimal precision. We obtain:

$$A = \begin{bmatrix} 1.000 & -2.000 \\ 0.003 & 5.000 \end{bmatrix}, b = \begin{bmatrix} 3.020 \\ 3.000 \end{bmatrix} \tag{42}$$

and compute:

$$R_2 \leftarrow R_2 - 0.003 \cdot R_1 \tag{43}$$

$$\begin{bmatrix} 1.000 & -2.000 \\ 0 & 5.006 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.020 \\ 2.991 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4.214 \\ 0.597 \end{bmatrix} \tag{44}$$

c. We first compute the scale of each row. By definition (Page 170 of [KC]), $s_1 = 5$, $s_2 = 2$. Then the scaled pivots become:

$$\begin{aligned} \frac{|a_{11}|}{s_1} &= 0.003/5.000 = 6 \times 10^{-4} \\ \frac{|a_{21}|}{s_2} &= 1.000/2.000 = 0.500 \end{aligned} \tag{45}$$

since the second row has a larger scaled pivot, we should swap the two rows, and the result given by scaled pivoting is the same as that of part b.

d. The results of part a. are inaccurate due to magnification of round-off error. In general, we should avoid division by a small number.

3. We first need to examine the values of ϵ for which the matrix is invertible. The determinant of matrix A is:

$$\det(A) = -1 - \epsilon^2 \le -1 \tag{46}$$

which is bounded away from zero. Therefore all choices of $\epsilon > 0$ is valid for nonsingular A. The inverse of A is readily found for a 2×2 matrix:

$$A^{-1} = -\frac{1}{(1+\epsilon^2)} \begin{bmatrix} 1-\epsilon & -2 \\ -1 & 1+\epsilon \end{bmatrix} \tag{47}$$

Therefore:

$$\begin{aligned} \|A\|_\infty &= \max\left(3+\epsilon, |1-\epsilon| + 1\right) \\ \left\|A^{-1}\right\|_\infty &= \max\left(2+\epsilon, 2 + \frac{|1-\epsilon|}{1+\epsilon^2}\right) \end{aligned} \tag{48}$$

The condition number represents the factor by which the worst-case relative error in numerical solution (e.g. round-off error) will be scaled.

First suppose $\epsilon \le 1$, then we have $|1-\epsilon| = 1-\epsilon$, $\|A\| = 3+\epsilon$, the value of $\left\|A^{-1}\right\|$ further depends on range of ϵ. We have:

$$x = \frac{1-x}{1+x^2} \Rightarrow x \approx 0.4534 \tag{49}$$

let ϵ^* be the root of the above equation. For $0 < \epsilon \le \epsilon^*$, we have $\left\|A^{-1}\right\| = 2 + \frac{1-\epsilon}{1+\epsilon^2}$, and otherwise $\epsilon^* < \epsilon \le 1$ yields $\left\|A^{-1}\right\| = 2+\epsilon$. By analyzing the behavior at boundaries, we have:

$$\begin{aligned} \lim_{\epsilon \to 0^+} \kappa(A;\epsilon) &= 9 \\ \lim_{\epsilon \to 1^-} \kappa(A;\epsilon) &= 12 \end{aligned} \tag{50}$$

Now we discuss the case when $\epsilon > 1$, upon which $|1-\epsilon| = \epsilon - 1$. We have $\|A\| = 1+\epsilon$, $\left\|A^{-1}\right\| = 2+\epsilon$, then $\kappa(A;\epsilon) = (1+\epsilon)(2+\epsilon) \sim O(\epsilon^2)$. The answer to this question is rather heuristic; full credits will be given for comprehensive consideration of different cases of values of $\epsilon > 0$. It is possible to ask for which values of ϵ, the best conditioning is achieved (by minimizing $\kappa(A;\epsilon)$ over ϵ), which occurs precisely at ϵ^*, with a condition number around $\kappa(A;\epsilon^*) \approx 8.473$.

4. See uploaded Jupyter notebook.

Math 211- Autumn 2023 – Homework 5 Solutions

1. We have the following linear system:

$$A = \begin{bmatrix} 6 & 3 & -2 \\ 1 & 3 & 0 \\ -3 & 1 & 5 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 12 \\ -6 \end{bmatrix}, x_0 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \tag{51}$$

a. By explicit calculations,

$$A^{-1} = \frac{1}{11}\begin{bmatrix} 3 & -\frac{17}{5} & \frac{6}{5} \\ -1 & \frac{24}{5} & -\frac{2}{5} \\ 2 & -3 & 3 \end{bmatrix} \tag{52}$$

$$\kappa_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = 11 \cdot \frac{1}{11} \cdot 8 = 8 \tag{53}$$

b. We have the splitting for Jacobi:

$$A = L + D + U \tag{54}$$

where L, U are the strict lower and upper diagonal parts of A, D is the diagonal part. The update rule is:

$$x^{(k+1)} = -D^{-1}(L+U)x^{(k)} + D^{-1}b \tag{55}$$

starting at x_0, we have:

$$x_1 = \begin{bmatrix} -1.167 \\ 3.667 \\ -1.000 \end{bmatrix}, x_2 = \begin{bmatrix} -1.667 \\ 4.389 \\ -2.633 \end{bmatrix}, x_3 = \begin{bmatrix} -2.572 \\ 4.556 \\ -3.078 \end{bmatrix} \tag{56}$$

c. In Gauss-Seidel, the update rule becomes:

$$x^{(k+1)} = -(D+L)^{-1}Ux^{(k)} + (D+L)^{-1}b \tag{57}$$

$$x_1 = \begin{bmatrix} -1.167 \\ 4.389 \\ -2.778 \end{bmatrix}, x_2 = \begin{bmatrix} -2.620 \\ 4.873 \\ -3.747 \end{bmatrix}, x_3 = \begin{bmatrix} -3.186 \\ 5.062 \\ -4.124 \end{bmatrix} \tag{58}$$

d. For Richardson, the update rule is:

$$x^{(k+1)} = (I - A)x^{(k)} + b \tag{59}$$

The method diverges:

$$x_1 = \begin{bmatrix} -12 \\ 7 \\ 3 \end{bmatrix}, x_2 = \begin{bmatrix} 48 \\ 10 \\ -61 \end{bmatrix}, x_3 = \begin{bmatrix} -389 \\ -56 \\ 372 \end{bmatrix} \tag{60}$$

e. Convergence is not guaranteed. Recall that for a method to converge, the iteration matrix must be a contractive linear map, namely, let B be an iteration matrix, c be a residual vector:

$$x^{(k+1)} = Bx^{(k)} + c \tag{61}$$

which corresponds to a fixed point x^* (if it exists):

$$x^* = Bx^* + c \tag{62}$$

then we define $e^{(k)} = x^{(k)} - x^*$, and comparing these two equations yields:

$$e^{(k+1)} = Be^{(k)} \Rightarrow \left\|e^{(k+1)}\right\| \leq \|B\| \cdot \left\|e^{(k)}\right\| \tag{63}$$

Therefore, in order to achieve convergence, we must have $\|B\| < 1$, which is not satisfied in our case for $(I - A)$, in Richardson.

2. We prove the bounds from left to right, first we investigate the relationship between 2-norm and ∞-norm.

   For all $x \in \mathbb{R}^n$, we have $\|Ax\|_\infty \geq n^{-1/2}\|Ax\|_2$, and $\|x\|_\infty \leq \|x\|_2$ implies $1/\|x\|_\infty \geq 1/\|x\|_2$. Then we have for any $x \neq 0$ that:

$$\frac{\|Ax\|_\infty}{\|x\|_\infty} \geq n^{-1/2}\frac{\|Ax\|_2}{\|x\|_2} \tag{64}$$

   We take supremum on the right hand side and obtain the first inequality.

   By similar reasoning, $\|Ax\|_2 \geq \|Ax\|_\infty$, and $\|x\|_2 \leq \sqrt{n}\|x\|_\infty$ implies $1/\|x\|_2 \geq n^{-1/2}/\|x\|_\infty$, together implying the second desired upper bound, upon taking supremum.

   The last two inequalities are omitted, where one may find the following bound useful:

$$\|x\|_1 \leq n\|x\|_2, \|x\|_2 \leq \sqrt{n}\|x\|_1 \tag{65}$$

3. We have Gauss-Seidel's iteration matrix of size $n \times n$:

$$-(D+L)^{-1}U = \begin{bmatrix} 0 & 2^{-1} & 0 & \cdots & 0 \\ 0 & 2^{-2} & 2^{-1} & \cdots & 0 \\ 0 & \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 2^{-1} \\ 0 & 2^{-n} & 2^{-n+1} & \cdots & 2^{-2} \end{bmatrix} \tag{66}$$

4. See attached code file.

Math 211- Autumn 2023 – Homework 6 Solutions

1. We recall that Richardson has the iteration matrix $(I - A)$. Therefore, $\|I - A\|_\infty < 1$ would be a necessary condition for convergence. By assumption, $a_{ii} = 1$ for all $1 \leq i \leq n$, then $\|I - A\|_\infty = \sum_{j\neq i} |a_{ij}| < 1$.

2. Consider matrices:
$$A = \begin{bmatrix} 5 & 3 & -1 \\ 2 & 6 & 4 \\ 1 & 2 & 10 \end{bmatrix}, B = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 5 & 1 \\ 1 & 2 & 10 \end{bmatrix} \tag{67}$$
   (a) By checking off-diagonal sum, A is not diagonally dominant (in the strict sense), whereas B is.

   (b) Both matrices are invertible (in particular, despite A not being diagonally dominant). Direct computation shows:
$$\kappa(A) \approx 6.196, \kappa(B) \approx 4.931 \tag{68}$$
   (c) Recall the update rule for Gauss-Seidel:
$$(D + L)x^{(k+1)} = -Ux^{(k)} + b \tag{69}$$
   which yields the iteration matrix:
$$-(D + L)^{-1}U \tag{70}$$
   then it is enough to compute the matrix explicitly for A and B and investigate the ∞-norm. For matrix A, we have:
$$\left\|(D + L)^{-1}U\right\| \approx 0.933 < 1 \tag{71}$$
   whereas for B we have:
$$\left\|(D + L)^{-1}U\right\| \approx 0.667 < 1 \tag{72}$$
   implying that both matrices yield convergent Gauss-Seidel iterations.

3. To bound the spectral radius, it is enough to apply Gershgorin's theorem and bound every eigenvalue. We have that all eigenvalues of A must be contained in the union of the following discs:
$$D_1 = \{z : |z - 3| \leq 3\}, D_2 = \{z : |z - 4| \leq 4\}, D_3 = \{z : |z - 8| \leq 2\} \tag{73}$$
One may visualize the regions by drawing circles centered at $z = 3, 4, 8$, with radii $3, 4, 2$, and we see that the maximum allowable eigenvalue cannot have magnitude exceeding 10, or $\rho(A) \leq 10$.

4. (a) Newton's form specifies:
$$p_k = p_{k-1} + c_k \prod_{j<k}(x - x_j) \tag{74}$$
   such that the additional term vanishes at $x = x_0, \ldots, x_{k-1}$, where we precisely recover $p_{k-1}(x_j) = y_j, 1 \leq j \leq k - 1$. In addition, we set:
$$c_k = \frac{y_k - p_{k-1}(x_k)}{\prod_{j<k}(x_k - x_j)} \tag{75}$$
   such that $p_k(x_k) = y_k$.
   The Newton polynomial can also be written in terms of the divided differences:
$$p_k(x) = \sum_{j=0}^{k} f[x_0, x_1, \ldots, x_j] \prod_{i=0}^{k-1}(x - x_i) \tag{76}$$
   By writting the points into the divided differences table, we find the incremental coefficients:
$$f[x_0, x_1] = -61, f[x_0, x_1, x_2] = -31, f[x_0, x_1, x_2, x_3] = -29 \tag{77}$$
   and $f[x_0] = 232$.

(b) For a set of nodes $x_0, \dots, x_n$, the cardinal functions refer to the form:

$$l_i(x) = \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}, \qquad 0 \leq i, j \leq n \tag{78}$$

such a function have the property that:

$$l_i(x_j) = \delta_{ij} \Rightarrow p_n(x_j) = \sum_{i=0}^{n} c_i l_i(x_j) = 0 + 0 + \dots + c_j + \dots + 0 = c_j \tag{79}$$

and thus allows us to recover exactly points y_j by setting the coefficients in the series at $c_j = y_j$. For each x_i, it is sufficient to compute the following constants:

$$\begin{aligned} x_0 &: (-2-1)(-2-2)(-2-6) = -96 \\ x_1 &: (1+2)(1-2)(1-6) = 15 \\ x_2 &: (2+2)(2-1)(2-6) = -16 \\ x_3 &: (6-2)(6-1)(6+2) = 160 \end{aligned} \tag{80}$$

5. See attached code file.

Math 211- Autumn 2023 – Homework 7 Solutions

1. For n points, the Newton form polynomial is of $(n-1)$-degree with the following form:

$$p_{n-1}(x) = \sum_{k=0}^{n-1} f[x_0, \dots, x_k] \prod_{j=0}^{k-1} (x - x_j) \tag{81}$$

(a) By using the divided differences table, we find:

$$f[x_0] = 2, f[x_0, x_1] = 1, f[x_0, x_1, x_2] = 3, f[x_0, x_1, x_2, x_3] = -8 \tag{82}$$

(b) The addition of a new point requires the additional computation of n divided differences. For instance, the addition of a fifth point $(x_4, y_4) = (6, 5)$ in our case requires us to compute:

$$f[x_3, x_4] = -71, f[x_2, x_3, x_4] = -\frac{124}{11}, f[x_1, x_2, x_3, x_4] = -\frac{37}{11} \tag{83}$$

which yields the new coefficient:

$$f[x_0, x_1, \dots, x_4] = 17/11 \tag{84}$$

2. (a) Following [KC], Section 6.3, we find the following coefficients:

$$f[x_0] = 2, f[x_0, x_1] = 1, f[x_0, x_1, x_1] = 8, f[x_0, x_1, x_1, x_1] = -5 \tag{85}$$

using the relations:

$$f[x_1] = -1, f[x_1, x_1] = f'(x_1) = -23, f[x_1, x_1, x_1] = \frac{1}{2} f''(x_1) = 23 \tag{86}$$

the Newton form interpolating polynomial is found to be:

$$p(x) = 2 + (x - x_0) + 8(x - x_0)(x - x_1) - 5(x - x_0)(x - x_1)^2 \tag{87}$$

(b) We have 2 points $x_0 = 1, x_1 = -2$, with up to second order information specified. Define the cardinal functions:

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{1}{3}(x + 3), l_1(x) = -\frac{1}{3}(x - 1) \tag{88}$$

we seek a form:

$$p(x) = f(x_0)A_0(x) + f(x_1)A_1(x) + f'(x_1)B_1(x) + f''(x_1)C_1(x) \tag{89}$$

such that:

$$\begin{cases} A_0(x_0) = 1, A_0(x_1) = 0 \\ A_1(x_0) = 0, A_1(x_1) = 1 \\ A_0'(x_0) = A_0'(x_1) = 0 \\ A_0''(x_0) = A_0''(x_1) = 0 \end{cases}, \begin{cases} B_1(x_0) = B_1(x_1) = 0 \\ B_1'(x_0) = 0, B_1'(x_1) = 1 \\ B_1''(x_0) = B_1''(x_1) = 0 \end{cases}, \begin{cases} C_1(x_0) = C_1(x_1) = 0 \\ C_1'(x_0) = C_1'(x_1) = 0 \\ C_1''(x_0) = 0, C_1''(x_1) = 1 \end{cases} \tag{90}$$

Remark: Since second order derivative is also interpolated, the exact form that satisfies the above conditions requires additional derivation not covered in class/textbook. The problem was graded largely based on completion, with full credits given to either:

(1) Writing out the constraints up to second order, without calculating the polynomials.

(2) Applying the first derivative formula correctly.

(c) We have:

$$f(1) = 2, f(-2) = -1, f'(-2) = -23, f''(-2) = a \tag{91}$$

which yields the coefficients by following the divided differences table:

$$f[x_0] = 2, f[x_0, x_1] = 1, f[x_0, x_1, x_1] = 8, f[x_0, x_1, x_1, x_1] = \frac{16-a}{6} \tag{92}$$

as expected, changing the highest order information (varying a) does not require us to update all of the previously calculated coefficients in Newton's form. In particular, if $a = 16$, we find that it is possible to interpolate 4 constraints with only a quadratic polynomial.

3. We have:

$$f(x) = e^x + x^2, f(-1) = e^{-1} + 1, f(0) = 1, f(2) = e^2 + 4 \tag{93}$$

we consider the piecewise linear function:

$$S(x) = \begin{cases} S_0(x) = a_0x + b_0, x \in [-1, 0] \\ S_1(x) = a_1x + b_1, x \in [0, 2] \end{cases} \tag{94}$$

such that:

$$\begin{aligned} S_0(-1) &= e^{-1} + 1 \\ S_0(0) &= S_1(0) = 1 \\ S_1(2) &= e^2 + 4 \end{aligned} \tag{95}$$

from which we calculate:

$$\begin{aligned} a_0 &= -e^{-1}, b_0 = 1 \\ a_1 &= \frac{1}{2}(e^2 + 3), b_1 = 1 \end{aligned} \tag{96}$$

4. The provided piecewise function is not a cubic spline. We can see that by checking continuity for S, S', S'' at the knots. In particular, let S_0, S_1, S_2 denote, respectively, the spline function on interval $(-\infty, 2], [2, 3], [3, \infty]$. We have:

$$S_0(2) = a, S_1(2) = -1 \Rightarrow a = -1 \tag{97}$$

however,

$$S_0'(2) = 2a, S_1'(2) = 2 \Rightarrow a = 1 \tag{98}$$

which implies that the splines of S_0, S_1 cannot satisfy cubic spline conditions.

Math 211- Autumn 2023 – Homework 7 Solutions

1. For n points, the Newton form polynomial is of $(n-1)$-degree with the following form:

$$p_{n-1}(x) = \sum_{k=0}^{n-1} f[x_0, \dots, x_k] \prod_{j=0}^{k-1} (x - x_j) \tag{99}$$

(a) By using the divided differences table, we find:

$$f[x_0] = 2, f[x_0, x_1] = 1, f[x_0, x_1, x_2] = 3, f[x_0, x_1, x_2, x_3] = -8 \tag{100}$$

(b) The addition of a new point requires the additional computation of n divided differences. For instance, the addition of a fifth point $(x_4, y_4) = (6, 5)$ in our case requires us to compute:

$$f[x_3, x_4] = -71, f[x_2, x_3, x_4] = -\frac{124}{11}, f[x_1, x_2, x_3, x_4] = -\frac{37}{11} \tag{101}$$

which yields the new coefficient:

$$f[x_0, x_1, \dots, x_4] = 17/11 \tag{102}$$

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$$f[x_0] = 2, f[x_0, x_1] = 1, f[x_0, x_1, x_1] = 8, f[x_0, x_1, x_1, x_1] = -5 \tag{103}$$

using the relations:

$$f[x_1] = -1, f[x_1, x_1] = f'(x_1) = -23, f[x_1, x_1, x_1] = \frac{1}{2} f''(x_1) = 23 \tag{104}$$

the Newton form interpolating polynomial is found to be:

$$p(x) = 2 + (x - x_0) + 8(x - x_0)(x - x_1) - 5(x - x_0)(x - x_1)^2 \tag{105}$$

(b) We have 2 points $x_0 = 1, x_1 = -2$, with up to second order information specified. Define the cardinal functions:

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{1}{3}(x + 3), l_1(x) = -\frac{1}{3}(x - 1) \tag{106}$$

we seek a form:

$$p(x) = f(x_0)A_0(x) + f(x_1)A_1(x) + f'(x_1)B_1(x) + f''(x_1)C_1(x) \tag{107}$$

such that:

$$\begin{cases} A_0(x_0) = 1, A_0(x_1) = 0 \\ A_1(x_0) = 0, A_1(x_1) = 1 \\ A_0'(x_0) = A_0'(x_1) = 0 \\ A_0''(x_0) = A_0''(x_1) = 0 \end{cases}, \begin{cases} B_1(x_0) = B_1(x_1) = 0 \\ B_1'(x_0) = 0, B_1'(x_1) = 1 \\ B_1''(x_0) = B_1''(x_1) = 0 \end{cases}, \begin{cases} C_1(x_0) = C_1(x_1) = 0 \\ C_1'(x_0) = C_1'(x_1) = 0 \\ C_1''(x_0) = 0, C_1''(x_1) = 1 \end{cases} \tag{108}$$

Remark: Since second order derivative is also interpolated, the exact form that satisfies the above conditions requires additional derivation not covered in class/textbook. The problem was graded largely based on completion, with full credits given to either:

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(c) We have:

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as expected, changing the highest order information (varying a) does not require us to update all of the previously calculated coefficients in Newton's form. In particular, if $a = 16$, we find that it is possible to interpolate 4 constraints with only a quadratic polynomial.

3. We have:

$$f(x) = e^x + x^2, f(-1) = e^{-1} + 1, f(0) = 1, f(2) = e^2 + 4 \tag{111}$$

we consider the piecewise linear function:

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such that:

$$\begin{aligned} S_0(-1) &= e^{-1} + 1 \\ S_0(0) &= S_1(0) = 1 \\ S_1(2) &= e^2 + 4 \end{aligned} \tag{113}$$

from which we calculate:

$$\begin{aligned} a_0 &= -e^{-1}, b_0 = 1 \\ a_1 &= \frac{1}{2}(e^2 + 3), b_1 = 1 \end{aligned} \tag{114}$$

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$$S_0(2) = a, S_1(2) = -1 \Rightarrow a = -1 \tag{115}$$

however,

$$S_0'(2) = 2a, S_1'(2) = 2 \Rightarrow a = 1 \tag{116}$$

which implies that the splines of S_0, S_1 cannot satisfy cubic spline conditions.